BRAID SEMISTATISTICS AND DOUBLY REGULAR R-MATRIX [*†]

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Abstract

We introduce "noninvertible" generalization of statistics - semistatistics replacing condition when double exchanging gives identity to "regularity" condition. Then in categorical language we correspondingly generalize braidings and the quantum Yang-Baxter equation. We define the doubly regular R-matrix and introduce obstructed regular bialgebras.

KEYWORDS : monoidal category, Yang-Baxter equation, semistatistics, braiding, obstruction, regularity, bialgebra, Hopf algebra

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Particle systems endowed with generalized statistics and its quantizations have been studied from different points of view (for review see e.g. [1, 2]). The color statistics have been considered in [3] (and refs. therein), and the category for color statistics has been described in details in [4]. The statistics in low dimensional spaces is based on the notion of the braid group [5, 6] (see also [7] for its acyclic extension). The construction for the category corresponding to a given triangular solution of the quantum Yang-Baxter equation has been given by Lyubashenko [8].The statistics corresponding for arbitrary triangular solution of the quantum Yang-Baxter equation called S-statistics has been discussed by Gurevich [9]. The mathematical formalism for the description of particle system with S-statistics is based on the theory of the tensor (monoidal) symmetric categories of MacLane [10]. The mathematical formalism related to an arbitrary braid statistics has been developed by Majid [11]. Such formalism is based on the concept of quasitensor (braided monoidal) categories which has been introduced by Joyal and Street [12].

The previous generalizations are "invertible" in the following sense: having the two-particle exchange process $12 \to 21$ (which in the simplest case usually yields the phase factor ± 1 or general anyonic factor [13]), then double exchanging gives identity $12 \to 12$. Here we weaken this requirement by moving to nearest "noninvertible" generalization of statistics – "regularity" as follows (symbolically)

$$12 \xrightarrow{a} 21 \xrightarrow{b} 12 = 12 \xrightarrow{\mathrm{id}} 12 \text{ "invertibility"}, \tag{1}$$

$$12 \xrightarrow{a} 21 \xrightarrow{b} 12 \xrightarrow{a} 21 = 12 \xrightarrow{a} 21 \text{ "left regularity"}, \tag{2}$$

$$21 \xrightarrow{b} 12 \xrightarrow{a} 21 \xrightarrow{b} 12 = 21 \xrightarrow{b} 12 \text{ "right regularity"}. \tag{3}$$

In this consideration we can treat usual statistics as *one* morphism a, in other words, the representation of the morphism a (because b can be found from the "invertibility" condition (1) which is $a \circ b = \mathrm{id}$ symbolically) by various phase factors or elements of R-matrix. Here we introduce the more abstract concept of "semistatistics" as a *pair* of exchanging morphisms a and b satisfying the "regularity" conditions (2)–(3) (symbolically $a \circ b \circ a = a$, $b \circ a \circ b = b$). The general regularization procedure for different systems was previously studied in [14–17].

We also introduce the notion of braid semistatistics and corresponding generalization of the quantum Yang-Baxter equation.

1 Braid semistatistics and regular Yang-Baxter equation

Let $\mathfrak{C}$ be a directed graph with objects $\mathrm{Ob}\mathfrak{C}$ and arrows $\mathrm{Mor}\mathfrak{C}$ [18, 19]. An N-*regular cocycle* $(X_1, X_2 \ldots, f_1, f_2 \ldots)$ in $\mathfrak{C}$, $N = 1, 2, \ldots$, is a sequence of arrows

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{N-1}} X_N \xrightarrow{f_N} X_1, \tag{4}$$

such that

$$\begin{aligned} f_1 \circ f_N \circ \cdots \circ f_2 \circ f_1 = f_1, \\ f_2 \circ f_1 \circ \cdots \circ f_3 \circ f_2 = f_2, \\ \\ f_N \circ f_{N-1} \circ \cdots \circ f_1 \circ f_N = f_N. \end{aligned} \tag{5}$$

We define N *obstructors* by

$$\begin{aligned} e_{X_1}^{(N)} &:= f_N \circ \cdots \circ f_2 \circ f_1 \in \mathrm{End}(X_1), \\ e_{X_2}^{(N)} &:= f_1 \circ \cdots \circ f_3 \circ f_2 \in \mathrm{End}(X_2), \\ &\vdots \\ e_{X_N}^{(N)} &:= f_{N-1} \circ \cdots \circ f_1 \circ f_N \in \mathrm{End}(X_N). \end{aligned} \tag{6}$$

The correspondence $e_X^{(N)} : X_n \in \mathrm{Ob}\mathfrak{C} \mapsto e_{X_n}^{(N)} \in \mathrm{End}(X_n)$, $n = 1, 2, \ldots, N$, is called an N -regular cocycle *obstruction* on $(X_1, X_2, \ldots, X_N | f_1, f_2, \ldots, f_N)$ in $\mathfrak{C}$.

Let $\mathfrak{M}$ be a monoidal category [12, 19] which abstractly defines the braid statistics. An N -regular *obstructed monoidal category* $\mathfrak{M}_{obtr}^{(N)}$ can be defined as usual, but instead of the identity $\mathrm{id}_X \otimes \mathrm{id}_Y = \mathrm{id}_{X\otimes Y}$ we have an obstruction structure $e_X^{(N)} = \{e_{X_n}^{(N)} \in \mathrm{End}(X_n); N = 1, 2, ...\}$ satisfying the condition

$$e_{X_n \otimes Y_n}^{(N)} = e_{X_n}^{(N)} \otimes e_{Y_n}^{(N)} \tag{7}$$

for every two N -regular cocycles $(X_1, X_2, \ldots, X_N | f_1, f_2, \ldots, f_N)$ and $(Y_1, Y_2, \ldots, Y_N | g_1, g_2, \ldots, g_N)$.

In a monoidal category $\mathfrak{M}$ for any two objects $X, Y \in \mathrm{ob}\mathfrak{M}$ and the product $X \otimes Y$ one can define a natural isomorphism (“braiding” [12]) by $\mathrm{B}_{X,Y} : X \otimes Y \to Y \otimes X$ satisfying the *symmetry condition* (“invertibility”)

$$\mathrm{B}_{Y,X} \circ \mathrm{B}_{X,Y} = \mathrm{id}_{X\otimes Y} \tag{8}$$

which formally defines $\mathrm{B}_{Y,X} = \mathrm{B}^{-1}_{X,Y} : Y \otimes X \to X \otimes Y$. The simplest type of braiding is the usual transposition $\tau_{X,Y}(x \otimes y) = y \otimes x$, where $x \in X$, $y \in Y$. Nonsymmetric braidings in context of the noncommutative geometry were considered in [20, 21] (see also [7]). In the obstructed monoidal category $\mathfrak{M}^{(N)}_{obstr}$ we introduce a "regular" extension of the braidings as follows. Let $(X_1, X_2, \ldots, X_N | f_1, f_2, \ldots, f_N)$ and $(Y_1, Y_2, \ldots, Y_N | g_1, g_2, \ldots, g_N)$ are regular cocycles and $e^{(N)}_{X_n}$, $e^{(N)}_{Y_n}$ are corresponding obstructors, then we have two sets of monoidal products of N-regular cocycles $X_1 \otimes Y_1$, $X_2 \otimes Y_2$, $\ldots X_N \otimes Y_N$, $f_1 \otimes g_1$, $f_2 \otimes g_2$, $\ldots f_N \otimes g_N$, and $Y_1 \otimes X_1$, $Y_2 \otimes X_2$, $\ldots Y_N \otimes X_N$, $g_1 \otimes f_1$, $g_2 \otimes f_2$, $\ldots g_N \otimes f_N$, and the obstructors satisfy $e^{(N)}_{X_n} \otimes e^{(N)}_{Y_n} = e^{(N)}_{X_n \otimes Y_n}$.

An *N-regular ("vector") braiding* $\tilde{\mathrm{B}}^{(N)}$ is a set of ("n-component") maps

$$X_n \otimes Y_n \overset{B^{(N),n}_{X_n \otimes Y_n}}{\longrightarrow} Y_n \otimes X_n$$

such that the following diagram

$$\begin{array}{ccccccc}
X_1 \otimes Y_1 & \overset{f_1 \otimes g_1}{\longrightarrow} & X_2 \otimes Y_2 & \overset{f_2 \otimes g_2}{\longrightarrow} & \ldots & \longrightarrow & X_N \otimes Y_N \\
\mathrm{B}^{(N),n}_{X_n \otimes Y_n} \downarrow & & \mathrm{B}^{(N),n}_{X_n \otimes Y_n} \downarrow & & & & \mathrm{B}^{(N),n}_{X_n \otimes Y_n} \downarrow \\
Y_1 \otimes X_1 & \overset{g_1 \otimes f_1}{\longrightarrow} & Y_2 \otimes X_2 & \overset{g_2 \otimes f_2}{\longrightarrow} & \ldots & \longrightarrow & Y_N \otimes X_N
\end{array}$$

is commutative. Instead of the symmetry condition (8) we introduce the *generalized (1-star) inverse* N-regular braiding $\tilde{\mathrm{B}}^{*(N)}$ with components satisfying

$$\mathrm{B}^{(N),n}_{X_n \otimes Y_n} \circ \mathrm{B}^{*(N),n}_{X_n \otimes Y_n} \circ \mathrm{B}^{(N),n}_{X_n \otimes Y_n} = \mathrm{B}^{(N),n}_{X_n \otimes Y_n}, \tag{9}$$

where in general $\mathrm{B}^{*(N),n}_{X_n \otimes Y_n} \neq \mathrm{B}^{(N),n,-1}_{X_n \otimes Y_n}$. We call such a category a "regular" category [15, 16] to distinct from symmetric and "braided" categories [12, 19].

The prebraiding relations in a symmetric monoidal category are defined as [2, 6, 12]

$$\mathrm{B}_{X \otimes Y, Z} = \mathbf{B}^{\mathsf{R}}_{X,Z,Y} \circ \mathbf{B}^{\mathsf{L}}_{X,Y,Z}, \tag{10}$$

$$\mathrm{B}_{Z, X \otimes Y} = \mathbf{B}^{\mathsf{L}}_{X,Z,Y} \circ \mathbf{B}^{\mathsf{R}}_{X,Y,Z}, \tag{11}$$

$$\mathbf{B}^{\mathsf{L}}_{X,Y,Z} = \mathrm{id}_X \otimes \mathrm{B}_{Y,Z}, \tag{12}$$

$$\mathbf{B}^{\mathsf{R}}_{X,Y,Z} = \mathrm{B}_{X,Y} \otimes \mathrm{id}_Z, \tag{13}$$

and prebraidings $\mathrm{B}_{X \otimes Y, Z}$ and $\mathrm{B}_{Z, X \otimes Y}$ satisfy (for symmetric case) the "invertibility" property

$$\mathrm{B}^{-1}_{X \otimes Y, Z} \circ \mathrm{B}_{X \otimes Y, Z} = \mathrm{id}_{X \otimes Y \otimes Z},$$

where $\mathrm{B}^{-1}_{X\otimes Y,Z} = \mathrm{B}_{Z,X\otimes Y}$. In this notations the standard "invertible" quantum Yang-Baxter equation takes the form [6, 21]

$$\mathbf{B}^{\mathsf{R}}_{Y,Z,X} \circ \mathbf{B}^{\mathsf{L}}_{Y,X,Z} \circ \mathbf{B}^{\mathsf{R}}_{X,Y,Z} = \mathbf{B}^{\mathsf{L}}_{Z,X,Y} \circ \mathbf{B}^{\mathsf{R}}_{X,Z,Y} \circ \mathbf{B}^{\mathsf{L}}_{X,Y,Z}. \tag{14}$$

For "noninvertible" braidings satisfying regularity (9) in search of the analogs of the definitions (12)–(13) it is naturally to exploit the obstructors $e^{(N)}_{X_n}$ instead of identity id_{X_n} ($n = 1 \dots N$) which were introduced in [22, 23]. They are defined as self-mappings $e^{(N)}_{X_n} : X_n \to X_n$ satisfying closure conditions

$$e^{(1)}_{X_n} = \mathrm{id}_{X_n}, \tag{15}$$
$$e^{(2)}_{X_n} = g \circ f, \tag{16}$$
$$e^{(3)}_{X_n} = h \circ g \circ f, \tag{17}$$
$$\dots$$

where $g, h \dots$ are some morphisms (see [23] for details). Then using the following triple maps

$$\mathbf{T}^{(N),n\ \mathsf{L}}_{X_n,Y_n,Z_n} : X_n \otimes Y_n \otimes Z_n \to X_n \otimes Z_n \otimes Y_n,$$
$$\mathbf{T}^{(N),n\ \mathsf{R}}_{X_n,Y_n,Z_n} : X_n \otimes Y_n \otimes Z_n \to Y_n \otimes X_n \otimes Z_n$$

defined similarly to (12)–(13)

$$\mathbf{T}^{(N),n\ \mathsf{L}}_{X_n,Y_n,Z_n} = e^{(N)}_{X_n} \otimes \mathrm{B}^{(N),n}_{Y_n,Z_n}, \tag{18}$$
$$\mathbf{T}^{(N),n\ \mathsf{R}}_{X_n,Y_n,Z_n} = \mathrm{B}^{(N),n}_{X_n,Y_n} \otimes e^{(N)}_{Z_n}, \tag{19}$$

we weaken prebraiding construction (10)–(11) in the following way

$$\mathrm{P}^{(N),n}_{X_n\otimes Y_n,Z_n} = \mathbf{T}^{(N),n\ \mathsf{R}}_{X_n,Z_n,Y_n} \circ \mathbf{T}^{(N),n\ \mathsf{L}}_{X_n,Y_n,Z_n}, \tag{20}$$
$$\mathrm{P}^{(N),n}_{Z_n,X_n\otimes Y_n} = \mathbf{T}^{(N),n\ \mathsf{L}}_{X_n,Z_n,Y_n} \circ \mathbf{T}^{(N)\ \mathsf{R}}_{X_n,Y_n,Z_n}. \tag{21}$$

Thus the corresponding "noninvertible" analog of the Yang-Baxter equation (21) is the set of "component"equations

$$\mathbf{T}^{(N),n\ \mathsf{R}}_{Y_n,Z_n,X_n} \circ \mathbf{T}^{(N),n\ \mathsf{L}}_{Y_n,X_n,Z_n} \circ \mathbf{T}^{(N),n\ \mathsf{R}}_{X_n,Y_n,Z_n} = \mathbf{T}^{(N),n\ \mathsf{L}}_{Z_n,X_n,Y_n} \circ \mathbf{T}^{(N),n\ \mathsf{R}}_{X_n,Z_n,Y_n} \circ \mathbf{T}^{(N),n\ \mathsf{L}}_{X_n,Y_n,Z_n}. \tag{22}$$

Its solutions can be found by application of the semigroup methods (see e.g. [24, 25]). Let us construct "braidings tower" of k-star regular braidings, and for 1-star regular braidings we have

$$\mathrm{P}^{(N),n}_{X_n\otimes Y_n,Z_n} \circ \overset{*}{\mathrm{P}}{}^{(N),n}_{X_n\otimes Y_n,Z_n} \circ \mathrm{P}^{(N),n}_{X_n\otimes Y_n,Z_n} = \mathrm{P}^{(N),n}_{X_n\otimes Y_n,Z_n} \tag{23}$$

$$\overset{*}{\mathrm{P}}{}^{(N),n}_{X_n\otimes Y_n,Z_n} \circ \mathrm{P}^{(N),n}_{X_n\otimes Y_n,Z_n} \circ \overset{*}{\mathrm{P}}{}^{(N),n}_{X_n\otimes Y_n,Z_n} = \overset{*}{\mathrm{P}}{}^{(N),n}_{X_n\otimes Y_n,Z_n}, \tag{24}$$

$$\mathrm{P}^{(N),n}_{Z_n,X_n\otimes Y_n} \circ \overset{*}{\mathrm{P}}{}^{(N),n}_{Z_n,X_n\otimes Y_n} \circ \mathrm{P}^{(N),n}_{Z_n,X_n\otimes Y_n} = \mathrm{P}^{(N),n}_{Z_n,X_n\otimes Y_n}, \tag{25}$$

$$\overset{*}{\mathrm{P}}{}^{(N),n}_{Z_n,X_n\otimes Y_n} \circ \mathrm{P}^{(N),n}_{Z_n,X_n\otimes Y_n} \circ \overset{*}{\mathrm{P}}{}^{(N),n}_{Z_n,X_n\otimes Y_n} = \overset{*}{\mathrm{P}}{}^{(N),n}_{Z_n,X_n\otimes Y_n}, \tag{26}$$

where $\overset{*}{\mathrm{P}}{}^{(N),n}_{X_n\otimes Y_n,Z_n}$is the generalized inverse (see e.g. [26]) for $\mathrm{P}^{(N),n}_{X_n\otimes Y_n,Z_n}$, and in general case $\overset{*}{\mathrm{P}}{}^{(N),n}_{X_n\otimes Y_n,Z_n} \neq \mathrm{P}^{(N),n,\,-1}_{X_n\otimes Y_n,Z_n}$. In a similar we can define k-star braidings $\mathrm{P}^{(N),n,\overbrace{**\ldots*}^{k}}_{X_n\otimes Y_n,Z_n}$ ($K \times N$-regular morphisms, their number is KN), where $k = 0, 1, 2 \ldots K-1$ [17, 22].

2 Regular Yang-Baxter operators

Let we have a set of regular obstructed algebras $\left(A_n, m_n, e^{(N)}_{A_n}\right)$ with multiplication m_n and obstructor $e^{(N)}_{A_n} : A_n \to A_n$ (see (6)) such that the diagram

$$\begin{array}{ccccccc}
A_1 \otimes A_1 & \overset{f_1\otimes f_1}{\to} & A_2 \otimes A_2 & \overset{f_2\otimes f_2}{\to} & \ldots & \to & A_N \otimes A_N \\
m_1 \downarrow & & m_2 \downarrow & & & & m_N \downarrow \\
A_1 & \overset{f_1}{\to} & A_2 & \overset{f_2}{\to} & \ldots & \to & A_N
\end{array}$$

is commutative, or

$$e^{(N)}_{A_n} \circ m_n = m_n \circ e^{(N)}_{A_n\otimes A_n}. \tag{27}$$

We introduce N Yang-Baxter operators $R^{(N)}_n : A_n \otimes A_n \to A_n \otimes A_n$ which commute with obstructors

$$R^{(N)}_n \circ e^{(N)}_{A_n\otimes A_n} = e^{(N)}_{A_n\otimes A_n} \circ R^{(N)}_n \tag{28}$$

and satisfy N-regular analog of the Yang-Baxter equation (set of N equations)

$$\left(e_{A_n}^{(N)} \otimes R_n^{(N)}\right) \circ \left(R_n^{(N)} \otimes e_{A_n}^{(N)}\right) \circ \left(e_{A_n}^{(N)} \otimes R_n^{(N)}\right) \tag{29}$$
$$= \left(R_n^{(N)} \otimes e_{A_n}^{(N)}\right) \circ \left(e_{A_n}^{(N)} \otimes R_n^{(N)}\right) \circ \left(R_n^{(N)} \otimes e_{A_n}^{(N)}\right).$$

We define 1-star N-regular obstructed Yang-Baxter operator (set of N operators $R_n^{(N)*}$) by

$$R_n^{(N)} \circ R_n^{(N)*} \circ R_n^{(N)} = R_n^{(N)}, \tag{30}$$
$$R_n^{(N)*} \circ R_n^{(N)} \circ R_n^{(N)*} = R_n^{(N)*}. \tag{31}$$

Similarly, one can define k-star operators $R_n^{\overbrace{**\cdots*}^{k}}$ ($K \times N$-regular Yang-Baxter operators, their number is KN), where $n = 1, 2 \ldots N$; $k = 0, 1, 2 \ldots K - 1$ [22, 23, 27].

3 Bialgebras and universal R-matrix

An obstructed (see [15]) N-regular bialgebra can be defined as a set of N bialgebras $\left(H_n, m_n, \Delta_n, e_{H_n}^{(N)}\right)$, where H_n, $(n = 1 \ldots N)$ are linear vector spaces over $\mathbb{K}$ with multiplications $m_n : H_n \otimes H_n \to H_n$ and comultiplications $\Delta_n : H_n \to H_n \otimes H_n$, but instead of identity map we have now N obstructors $e_{H_n}^{(N)} : H_n \to H_n$ (analogies of mappings (6)) satisfying the consistency conditions

$$e_{H_n}^{(N)} \circ m_n = m_n \circ e_{H_n \otimes H_n}^{(N)}, \qquad \Delta_n \circ e_{H_n}^{(N)} = e_{H_n \otimes H_n}^{(N)} \circ \Delta_n. \tag{32}$$

The associativity and coassociativity now have the form

$$m_n \circ \left(m_n \otimes e_{H_n}^{(N)}\right) = m_n \circ \left(e_{H_n}^{(N)} \otimes m_n\right), \quad \left(\Delta_n \otimes e_{H_n}^{(N)}\right) \circ \Delta_n = \left(e_{H_n}^{(N)} \otimes \Delta_n\right) \circ \Delta_n.$$

The Yang-Baxter operators $R_n^{(N)} : H_n \otimes H_n \to H_n \otimes H_n$ also satisfy the additional consistency conditions (analogy of (28))

$$e_{H_n \otimes H_n}^{(N)} \circ R_n^{(N)} = R_n^{(N)} \circ e_{H_n \otimes H_n}^{(N)}$$

and the set of N Yang-Baxter equations of type (29), as follows

$$\left(e_{H_n}^{(N)} \otimes R_n^{(N)}\right) \circ \left(R_n^{(N)} \otimes e_{H_n}^{(N)}\right) \circ \left(e_{H_n}^{(N)} \otimes R_n^{(N)}\right) \tag{33}$$
$$= \left(R_n^{(N)} \otimes e_{H_n}^{(N)}\right) \circ \left(e_{H_n}^{(N)} \otimes R_n^{(N)}\right) \circ \left(R_n^{(N)} \otimes e_{H_n}^{(N)}\right),$$

which defines the universal obstructed N-regular R-matrix for obstructed N-regular bialgebra $\left(H_n, m_n, \Delta_n, e_{H_n}^{(N)}\right)$. We define 1-star universal obstructed N-regular R-matrix by

$$R_n^{(N)} \circ R_n^{(N)*} \circ R_n^{(N)} = R_n^{(N)}, \qquad R_n^{(N)*} \circ R_n^{(N)} \circ R_n^{(N)*} = R_n^{(N)*}. \tag{34}$$

As above one can define k-star Yang-Baxter operators $R_n^{\overbrace{* * \cdots *}^{k}}$ (set of KN operators) $n = 1, 2 \ldots N; k = 0, 1, 2 \ldots K - 1$ [22, 23]. Then the convolution product can be defined (in "components") as

$$s \star_n t := m_n \circ (s \otimes t) \circ \triangle_n, \tag{35}$$

where $s, t \in \hom_{m_n}(H_n, H_n)$.

Let A be an N-regular obstructed algebra with N obstructors $e_A^{(N)}$ and multiplication m, and $R^{(N)}$ be an N-regular Yang-Baxter operator on A, then the algebra A with the multiplication $m_R = m \circ R^{(N)}$ is also an N-regular obstructed algebra. Indeed, from definition (27) we have $e_A^{(N)} \circ m = m \circ e_{A\otimes A}^{(N)}$, and then from (28) we obtain $m_R \circ e_{A\otimes A}^{(N)} = m \circ R^{(N)} \circ e_{A\otimes A}^{(N)} = m \circ e_{A\otimes A}^{(N)} \circ R^{(N)} = e_A^{(N)} \circ m \circ R^{(N)} = e_A^{(N)} \circ m_R$.

Let C be an N-regular obstructed coalgebra with N obstructors $e_C^{(N)}$ and comultiplication Δ, and $R^{(N)}$ be an N-regular Yang-Baxter operator on C, then the algebra C with the comultiplication $\Delta_R = R^{(N)} \circ \Delta$ is also an N-regular obstructed coalgebra. Indeed, from definition (32) we have $\Delta \circ e_A^{(N)} = e_{A\otimes A}^{(N)} \circ \Delta$, and then from (28) we obtain $\Delta_R \circ e_A^{(N)} = R^{(N)} \circ \Delta \circ e_A^{(N)} = R^{(N)} \circ e_{A\otimes A}^{(N)} \circ \Delta = e_{A\otimes A}^{(N)} \circ R^{(N)} \circ \Delta = e_{A\otimes A}^{(N)} \circ \Delta_R$.

4 Doubly regular Hopf algebras

Usual antipode is defined as inverse to the identity under convolution, if and only if there exist unit and counit for a bialgebra [28, 29]. Since we do not require existence of unit and counit in obstructed bialgebras, we have to define some more

general analog of antipode. The Von Neumann regular antipode for weal Hopf algebras was considered in [30–32] ("non-unital"/"nonsymmetric" antipodes were considered in [33]). By analogy we can introduce the obstructed N-regular antipode (set of N antipodes) for every bialgebra $\left(H_n, m_n, \Delta_n, e_{H_n}^{(N)}\right)$ as a generalized inverse for obstructor

$$e_{H_n}^{(N)} \star_n S_n^{(N)} \star_n e_{H_n}^{(N)} = e_{H_n}^{(N)}, \quad S_n^{(N)} \star_n e_{H_n}^{(N)} \star_n S_n^{(N)} = S_n^{(N)}. \tag{36}$$

In this way we define LN higher L-regular analogs of antipode $S_n^{\overbrace{**\ldots*}^{l}}$ ($l = 0, 1, 2 \ldots L - 1$), similarly to K-star regular quantities above. For example, in the case $l = 1$ we have instead of (36) the following set of defining equations

$$\begin{gathered} e_{H_n}^{(N)} \star_n S_n^{(N)} \star_n S_n^{(N)*} \star_n e_{H_n}^{(N)} = e_{H_n}^{(N)}, \\ S_n^{(N)} \star_n S_n^{(N)*} \star_n e_{H_n}^{(N)} \star_n S_n^{(N)} = S_n^{(N)}, \\ S_n^{(N)*} \star_n e_{H_n}^{(N)} \star_n S_n^{(N)} \star_n S_n^{(N)*} = S_n^{(N)*}. \end{gathered}$$

An obstructed N-regular bialgebra $\left(H_n, m_n, \Delta_n, e_{H_n}^{(N)}\right)$ with L-regular antipode is called obstructed $N \times L$-regular (doubly regular) Hopf algebra

$$\left(H_n, m_n, \Delta_n, e_{H_n}^{(N)}, S_n^{\overbrace{**\ldots*}^{l}}\right),$$

where $n = 1, 2 \ldots N$; $l = 0, 1, 2 \ldots L - 1$.

Note, that in general, obstructed $N \times L$-regular Hopf algebras

$$\left(H_n, m_n, \Delta_n, e_{H_n}^{(N)}, S_n^{\overbrace{**\ldots*}^{l}}\right)$$

do not contain unit and/or counit (analogously to [32, 33]).

In the opposite case it can be possible that for each $N \times L$-regular Hopf algebra

$$\left(H_n, m_n, \Delta_n, e_{H_n}^{(N)}, S_n^{\overbrace{**\ldots*}^{l}}\right)$$

there exist unit η_n and counit ε_n. If we have one antipode for every n, then it should satisfy

$$\left(S_n^{(N)} \otimes e_{H_n}^{(N)}\right) \circ \Delta_n = \left(e_{H_n}^{(N)} \otimes S_n^{(N)}\right) \circ \Delta_n = \eta_n \circ \varepsilon_n.$$

We call P_n, Q_n obstructed N-regular modules if for each n there exist maps $\rho_{P_n} : P_n \otimes H_n \to P_n$ and $\rho_{Q_n} : H_n \otimes Q_n \to Q_n$, such that

$$e_{P_n}^{(N)} \circ \rho_{P_n} = \rho_{P_n} \circ \left(e_{P_n}^{(N)} \otimes e_{H_n}^{(N)}\right), \quad \rho_{Q_n} \circ e_{Q_n}^{(N)} = \left(e_{H_n}^{(N)} \otimes e_{Q_n}^{(N)}\right) \circ \rho_{Q_n},$$

where $e_{P_n}^{(N)}$ and $e_{Q_n}^{(N)}$ are obstructors for modules P_n and Q_n (see (6)).

Let R_n be the universal obstructed N-regular R-matrix on the obstructed N-regular bialgebra $\left(H_n, m_n, \Delta_n, e_{H_n}^{(N)}\right)$, and P_n and Q_n are left modules over H_n, then there is obstructed N-regular braiding $B_{P_n,Q_n}^{(N)} : P_n \otimes Q_n \to Q_n \otimes P_n$, such that $B_{P_n,Q_n}^{(N)} (P_n \otimes Q_n) = \tau_{P_n,Q_n} \left(R_n^{(N)} (P_n \otimes Q_n)\right)$, where $R_n^{(N)}$ is the corresponding Yang-Baxter operator.

5 Conclusions

Thus, in this paper we have constructed a general categorical approach for systems endowing "noninvertible" ("regular") statistics (2)–(3) — semistatistics — using methods of [1]- [17]. We introduced doubly regular prebraiding and braiding and obtained the set of regular Yang-Baxter equations in terms of obstructors. The doubly regular Yang-Baxter operators, bialgebras and Hopf algebras are considered.

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